

Multi-pair states in electron–positron pair creation

Anton Wöller^{*}, Heiko Bauke[†] and Christoph H. Keitel

Max-Planck-Institut für Kernphysik, Saupfercheckweg 1, 69117 Heidelberg, Germany

(Dated: July 26, 2016)

Ultra strong electromagnetic fields can lead to spontaneous creation of single or multiple electron–positron pairs. A quantum field theoretical treatment of the pair creation process combined with numerical methods provides a description of the fermionic quantum field state, from which all observables of the multiple electron–positron pairs can be inferred. This allows to study the complex multi-particle dynamics of electron–positron pair creation in-depth, including multi-pair statistics as well as momentum distributions and spin. To illustrate the potential benefit of this approach, it is applied to the intermediate regime of pair creation between nonperturbative Schwinger pair creation and perturbative multiphoton pair creation where the creation of multi-pair states becomes nonnegligible but cascades do not yet set in. Furthermore, it is demonstrated how spin and helicity of the created electrons and positrons are affected by the polarization of the counterpropagating laser fields, which induce the creation of electron–positron pairs.

1. Introduction

Relativistic quantum field theory predicts the possibility of electron–positron pair creation from the vacuum in the presence of a strong electromagnetic field. Initiated by the pioneering works by Sauter, Heisenberg, and Euler [1, 2] many theoretical [3–14] and experimental [15–17] efforts have been undertaken to study pair creation; see Refs. [9, 18] for recent reviews. Spontaneous pair creation by static fields is expected to set in at the Schwinger critical field strength of $E_S = 1.3 \times 10^{18}$ V/m, which cannot be reached even by the strongest laser facilities available today. However, pair creation may be assisted by time- and space-dependent electromagnetic fields. Novel light sources envisage to provide field intensities in excess of 10^{20} W/cm² and field frequencies in the X-ray domain [19–23]. The ELI-Ultra High Field Facility, for example, aims to reach intensities exceeding even 10^{23} W/cm² corresponding to a field strength of about 10^{15} V/m, which may be sufficient to observe pair creation [24–30].

Pair creation in time-dependent electromagnetic fields can be characterized via the classical nonlinearity parameter $\xi = eE/(m_0 c \omega)$ as nonperturbative Schwinger pair creation ($\xi \gg 1$) or perturbative multiphoton pair creation ($\xi \ll 1$), where m_0 denotes the electron mass, c the speed of light, e the elementary charge, E the electric field’s peak strength, and ω its angular frequency. Both regimes are accessible by different analytical methods. Experimentally the nonperturbative Schwinger regime may be realized by high-intensity optical lasers and has attracted a considerable amount of theoretical research, e. g., predicting pair-creation cascades by semi-classical methods [31–38]. In this regime, pair-creation can be understood as a tunneling process [39, 40], which is exponentially suppressed for subcritical fields. In the other extreme regime which is relevant for weaker intensities but ultra high photon energies, perturbative multiphoton pair creation has been approached experimentally [16, 41]. New directions in pair production

have also been proposed [42–45] by combining both regimes leading to the dynamically assisted Schwinger effect.

The intermediate regime, i. e., nonperturbative multiphoton pair creation [46] with $\xi \approx 1$, however, is less studied [41, 47–51] and a comprehensible physical picture is missing. Despite this, many interesting phenomena may be expected like the coherent production of multiple pairs [4, 5, 52], quantum statistical influences [5, 53], Pauli exclusion effects [5, 54] and a mixture of signatures known from pure tunneling and multiphoton processes [10, 39, 45, 50]. This regime requires a quantum field theoretical approach [55] which accounts for possible multi-pair states as well as for the temporal and spatial variations of the electromagnetic environment. By applying the in-out formalism [56, 57] in combination with numerical solutions of the time-dependent Dirac equation [58–60], we will analyze the fermionic field state and its coherent quantum dynamics in the nonperturbative multiphoton regime. In comparison to other approaches, where in most cases only the dynamics of specific observables like the number of produced pairs is calculated, the evaluation of the fermionic field state allows to explore all amplitudes of different single and multi-pair states. These amplitudes become nontrivial for space- and time-dependent external fields, because in this case the fermionic field state is not just a product state, as is the case for external fields which depend only on time. Furthermore, all possible observables of interest can be calculated once the fermionic field state is known. We anticipate as well the experimental relevance as nonperturbative multiphoton pair creation may be observable employing laser sources of lower intensities [41].

2. Theoretical foundations

The pair creation probability is the central quantity in the study of pair creation. It is, however, a rather coarse-grained quantity. It does not distinguish whether only a single electron–positron pair has been created or if the electron–positron pair is part of a multi-particle state, which contains a larger number of electron–positron pairs. This fine-grained information is encoded in the quantum field state, into which an initial vacuum state evolves

^{*} woeller@mpi-hd.mpg.de

[†] heiko.bauke@mpi-hd.mpg.de

$$\begin{aligned}
|\text{out}\rangle = & C_v |0\rangle + \\
& c_1 \begin{array}{|c|} \hline \uparrow \downarrow \\ \hline -3 \quad 0 \end{array} + c_2 \begin{array}{|c|} \hline \uparrow \downarrow \\ \hline 2 \quad 0 \end{array} + c_3 \begin{array}{|c|} \hline \uparrow \downarrow \\ \hline -2 \quad 0 \end{array} + \dots \\
& c_4 \begin{array}{|c|} \hline \uparrow \downarrow \uparrow \downarrow \\ \hline -3 \quad 0 \quad 2 \quad 0 \end{array} + c_5 \begin{array}{|c|} \hline \uparrow \downarrow \uparrow \downarrow \\ \hline -3 \quad 0 \quad -2 \quad 0 \end{array} + c_6 \begin{array}{|c|} \hline \uparrow \downarrow \uparrow \downarrow \\ \hline 2 \quad 0 \quad -2 \quad 0 \end{array} + \dots
\end{aligned}$$

FIG. 1: The vacuum evolves under the effect of a strong electromagnetic field into a quantum field state $|\text{out}\rangle$, which is a quantum mechanical superposition of the vacuum state $|0\rangle$ (1st row), single-electron-positron-pair states (2nd row), states with two electron-positron pairs (3rd row), and further multi-pair states. Multi-pair states with two electrons or two positrons having the same quantum numbers are excluded by the Pauli principle (last state in 3rd row).

under the effect of a strong electromagnetic field. This state is a quantum mechanical superposition of the vacuum state and all permissible multi-electron-positron-pair states; see Fig. 1. In this pictorial representation, each pair's quantum numbers are represented by arrows indicating the spin and numbers specifying the momentum. The left arrow and number (blue color) belong to the electron and the right (red color) to the positron. Due to the electron's fermionic character, the quantum field state must be compatible with the Pauli exclusion principle, which forbids multi-pair states, where two electrons or two positrons have the same quantum numbers.

In strong-field quantum electrodynamics the potential of the electromagnetic field is split into a quantized radiation field and a classical background field. This leads to the Furry picture formulation of quantum electrodynamics [56, 57, 61], where the classical background field is treated exactly but the effect of the quantized radiation field calculated by means of perturbation theory. At the onset of pair creation, where the number of created pairs is small, it is justified to assume that the interaction between the quantized radiation field and the quantized matter field can be neglected completely. In this case, the evolution of a quantum field state in an external electromagnetic field is determined by the Hamiltonian

$$\hat{H}(t) = \int \hat{\psi}(\mathbf{r})^\dagger \mathcal{H}(\mathbf{r}, t) \hat{\psi}(\mathbf{r}) d^3r, \quad (1)$$

where $\hat{\psi}(\mathbf{r})$ denotes the time-independent spinor field operator and [62]

$$\mathcal{H}(\mathbf{r}, t) = \boldsymbol{\alpha} \cdot (-i\nabla - q\mathbf{A}(\mathbf{r}, t)) + m_0\beta \quad (2)$$

is the Hamiltonian density for a particle with charge q and rest mass m_0 . Furthermore, $\boldsymbol{\alpha} = (\alpha_x, \alpha_y, \alpha_z)^\top$ and β represent the Dirac matrices [63, 64]. Describing the electromagnetic field via an external vector potential $\mathbf{A}(\mathbf{r}, t)$ is justified as long as back reaction of the created particles on the electromagnetic environment can be neglected. The spinor field operator $\hat{\psi}(\mathbf{r})$ can be decomposed into two possible sets of orthonormal functions ${}_{\pm}\varphi_n(\mathbf{r})$ and ${}^{\pm}\varphi_n(\mathbf{r})$ fulfilling the boundary conditions

$$\mathcal{H}(\mathbf{r}, t_{\text{in}}) {}_{\pm}\varphi_n(\mathbf{r}) = \pm \varepsilon_n {}_{\pm}\varphi_n(\mathbf{r}), \quad (3a)$$

$$\mathcal{H}(\mathbf{r}, t_{\text{out}}) {}^{\pm}\varphi_n(\mathbf{r}) = \pm \varepsilon_n {}^{\pm}\varphi_n(\mathbf{r}), \quad (3b)$$

where ${}_{+}\varepsilon_n > 0$, ${}_{-}\varepsilon_n < 0$ and ${}^{+}\varepsilon_n > 0$, ${}^{-}\varepsilon_n < 0$ denote the corresponding positive and negative eigenenergies, with n labeling

the quantum state. The times t_{in} and t_{out} specify the begin and the end of the interaction with an external electromagnetic field, which vanishes outside the interval $[t_{\text{in}}, t_{\text{out}}]$. The modes ${}_{\pm}\varphi_n(\mathbf{r})$ and ${}^{\pm}\varphi_n(\mathbf{r})$ are chosen as free-particle states with definite momentum, energy, and spin orientation (in the z direction) at t_{in} and t_{out} . Thus, in our notation the quantum number n gathers the momentum quantum number and the spin quantum number.

The dynamics of the quantum field state can be obtained by propagating all basis vectors of the Hilbert space via the Dirac equation [7, 65, 66] while including the external electromagnetic fields with their full time and space dependence [13]. Computing numerically the time propagation of the modes ${}_{\pm}\varphi_n(\mathbf{r}, t)$ yields the matrices $G^{(+)}_{-}$ and $G^{(-)}_{-}$ with the elements

$$G^{(\pm)}_{-mn} = {}_{\pm}\varphi_m(\mathbf{r})^\dagger G(\mathbf{r}, t_{\text{out}}; \mathbf{r}', t_{\text{in}}) {}_{\pm}\varphi_n(\mathbf{r}') d^3r' d^3r. \quad (4)$$

As shown in Ref. [57], the relative probability amplitude for creating one pair composed by an electron with quantum number m and a positron with quantum number n is given by

$$\omega(m\bar{n}|0) = -[G^{(+)}_{-} G^{(-)}_{-}]_{mn} \quad (5)$$

and the vacuum-to-vacuum probability by

$$|C_v|^2 = |\det G^{(-)}_{-}|^2. \quad (6)$$

Using these quantities, the fermionic field state (including all possible multi-pair states) at time t_{out} can be written as [67]

$$|\text{out}\rangle = C_v \sum_{N\{m,n\}} \frac{1}{N!} \left(\prod_{i=1}^N \omega(m_i^+ \bar{n}_i^- | 0) \right) |N_{\{m,n\}}\rangle. \quad (7)$$

Here, the quantum field state with N pairs

$$|N_{\{m,n\}}\rangle \equiv \hat{b}_{n_1}^\dagger \dots \hat{b}_{n_N}^\dagger \hat{a}_{m_1}^\dagger \dots \hat{a}_{m_1}^\dagger |0\rangle \quad (8)$$

is defined in terms of the creation operators for the electron $\hat{a}_{m_i}^\dagger$ and the positron $\hat{b}_{n_i}^\dagger$ (with quantum numbers indicated by the index sets $\{m\}$ and $\{n\}$), and the vacuum state $|0\rangle$. Note that the definition of $|N_{\{m,n\}}\rangle$ incorporates the Pauli exclusion principle, because a fermionic creation operator with a specific set of quantum numbers acting twice on some state yields zero. The corresponding multi-pair state's probability amplitude is

$$c_{\{m,n\}} = \langle N_{\{m,n\}} | \text{out} \rangle. \quad (9)$$

Accordingly, the probability that the final state $|\text{out}\rangle$ contains N pairs becomes

$$c_N = \langle \text{out} | \left[\frac{1}{N!} \sum_{\{m,n\}} |N_{\{m,n\}}\rangle \langle N_{\{m,n\}}| \right] | \text{out} \rangle. \quad (10)$$

Various observables can be inferred from (9) and (10). For example, the total electron spin and the total positron spin averaged over all multi-pair states with N pairs and similarly the average total helicity may be defined respectively, as

$$s_N^\pm = \frac{1}{c_N} \langle \text{out} | \left[\frac{1}{N!} \sum_{\{m,n\}} |N_{\{m,n\}}\rangle \left(\sum_i^N s_i^\pm \right) \langle N_{\{m,n\}}| \right] | \text{out} \rangle, \quad (11)$$

$$h_N^\pm = \frac{1}{c_N} \langle \text{out} | \left(\frac{1}{N!} \sum_{\{m,n\}} |N_{\{m,n\}}\rangle \left(\sum_i^N b_i^\pm \right) \langle N_{\{m,n\}} | \right) | \text{out} \rangle. \quad (12)$$

The index “ \pm ” distinguishes between electrons (“+”) and positrons (“−”) again, and s_i^\pm and b_i^\pm designate the expectation values of the z -component of the spin operator $\hat{\sigma}_z$ and the helicity operator $\hat{h} = \hat{\sigma} \cdot \hat{p}/|\hat{p}|$ of the indexed single-pair state, with \hat{p} denoting the momentum operator. For example, s_i^- corresponds to the spin of the i th positron with the quantum number m_i .

3. Elliptically polarized light beams

The theoretical formalism of pair creation as outlined above is not specific for a particular external-field configuration. In the following, we consider two counterpropagating monochromatic laser fields with wavelength λ and wave vectors equal to $\mathbf{k}_\pm = \pm k \mathbf{e}_z$ where $k = 2\pi/\lambda$. The electromagnetic field is parametrized in terms of the Jones vectors [68] $|l\rangle = (\mathbf{e}_x + i\mathbf{e}_y)/\sqrt{2}$ and $|r\rangle = (\mathbf{e}_x - i\mathbf{e}_y)/\sqrt{2}$, which correspond to circular left and circular right polarization, respectively. The electric field component of each laser is given by

$$\mathbf{E}_\pm(\mathbf{r}, t) = \text{Re} \left(E (\cos \alpha_\pm |l\rangle + \sin \alpha_\pm |r\rangle) e^{i(\pm k z - \omega t)} \right). \quad (13)$$

The magnetic field component follows via $\mathbf{B}_\pm = \mathbf{k}_\pm \times \mathbf{E}_\pm/k$, and the mean intensity equals $\varepsilon_0 E^2/2$. Both lasers’ polarizations are determined by α_\pm . Here, α_\pm ranges from 0 to $\pi/2$, where 0 corresponds to circular left polarization (photonic spin “down”), $\pi/4$ to linear polarization, and $\pi/2$ to circular right polarization (photonic spin “up”). For simplicity, we assume that both plane waves have the same degree of ellipticity, which requires $\alpha_+ = \alpha_-$ or $\alpha_+ = \pi/2 - \alpha_-$. The helicities of both beams are opposite to each other in the former case, whereas they are equal in the latter. The turn-on and turn-off of the external field is realized by modulating the vector potential by a window function with \sin^2 -shaped turn-on/off of length ΔT and a flat plateau of length T in between. Observables are always determined after the electromagnetic field has been turned off, which is necessary for a clear physical interpretation [13, 27, 47].

When the quantum field state (7) shall be determined by numerical means, it is necessary to truncate the quantities $G^{(+)}|_-$ and $G^{(-)}|_-$ to finite matrices. Due to the monochromatic nature of the counterpropagating waves only quantum field modes separated by a multiple of \mathbf{k}_\pm couple to each other. Thus, the Hilbert space separates into independent subspaces [13]. Each of the momentum subspaces consists of the modes with momentum $n\mathbf{k}_\pm + \mathbf{k}_0$, where n runs over all integers and \mathbf{k}_0 identifies the subspace’s momentum origin. All parts of the fermionic field state in Eq. (7) corresponding to a specific subspace can be calculated independently from others. Furthermore, every single subspace can be truncated to a finite number of modes due to the fact that the coupling between different modes becomes very weak with increasing momentum. For simplicity, we will concentrate in the following on the subspace with $\mathbf{k}_0 = 0$, which maximizes the pair production probability.

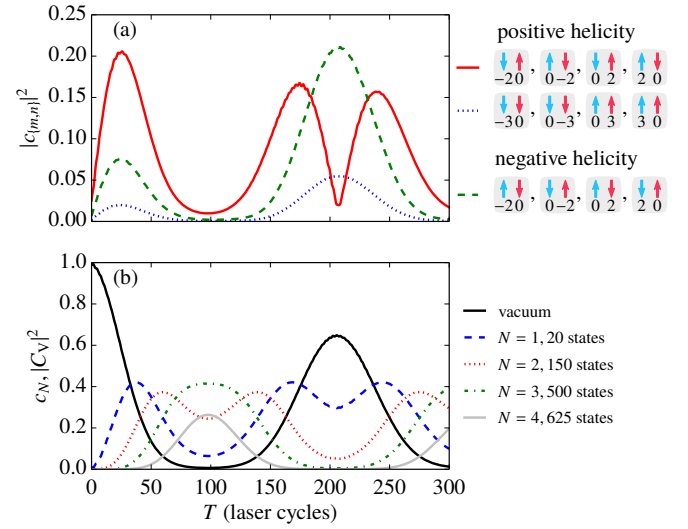


FIG. 2: (a) Evolution (corresponding to different total interaction periods) of single-pair states created in two counterpropagating laser beams of the total interaction time. Due to symmetry reasons, there are always four different single-pairs with the same probability $|c_{m,n}|^2$. (b) Evolution of the initial vacuum state and its decomposition into different multi-pair states. Initially, the states with lower pair count contribute most, but are outweighed later by states with higher pair count due to the high number of possible multi-pair states. Laser field parameters are $\omega = 0.746m_0$, $E = 4.9 \times 10^{17}$ V/m, and $\alpha_+ = 0.2\pi/4$ and both beams have the same nonzero helicity. Note that the electromagnetic field has been turned off before the probabilities $|c_{m,n}|^2$, c_N , and $|C_V|^2$ are determined.

4. Multi-pair states dynamics and numerical results

Equation (9) provides the probability amplitudes for states with a fixed number N of created pairs and specific sets of quantum numbers. The resulting probabilities of the fundamental single-pair states are presented in Fig. 2(a). Our numerical results show that only 20 single-pair states with nonnegligible weight occur for the chosen laser field parameters and 12 of them are exemplarily displayed. Note that all electrons/positrons with nonzero momentum have the same helicity within each group. The oscillatory behavior of the probability $|c_{m,n}|^2$ shown in Fig. 2(a), which resembles Rabi flopping in atomic physics, indicates that not only transitions from negative-energy states to positive-energy states occurs but also the inverse process is of importance.

The final quantum field state also contains a certain number of multi-pair states with nonnegligible weight, which all consist of the 20 fundamental single-pair states. The probability to find a multi-pair state with N pairs c_N and the vacuum probability $|C_V|^2$ are shown in Fig. 2(b). For zero interaction time, the resulting state is in the vacuum state with probability one. Extending the interaction time results in a non-trivial dependence for the probability of the final state to be in a single- or multi-pair state. For short interaction times, it is most probable to find a single-pair state and probabilities for multi-pair states

decrease with the number of included pairs. This is no longer true for longer interaction times. After an interaction time of about 100 laser cycles, for example, multi-pair states with 3 or 4 pairs are more probable than states with 1 or 2 pairs.

This can be explained by combinatorics and the specific values of the relative probability amplitudes $\omega(\vec{m}\vec{n}|0)$. The probability c_N is the sum over $|c_{(m,n)}|^2$ of all N -pair states. The coefficient $|c_{(m,n)}|^2$ for an N -pair state is always smaller than a coefficient $|c'_{(m,n)}|^2$ for some N' -pair state, if the N -pair state includes all pairs of the N' -pair state (i. e., $N' < N$) and all $|\omega(\vec{m}\vec{n}|0)|^2 < 1$. The coefficient c_N , however, can be larger than $c_{N'}$ because the number of possible combinations of single-pair states into multi-pair states grows exponentially with the number of pairs. It is only limited by the Pauli principle, which forbids to combine single-pair states with equal quantum numbers for either the electrons or the positrons into a multi-pair state. Furthermore, if $|\omega(\vec{m}\vec{n}|0)|^2 > 1$ for two single-pair states, then their combination has an amplitude that is larger than each of the single pairs' amplitudes. Nevertheless, the resulting probability does not exceed unity because the amplitude is normalized by C_N ; see Eq. (7).

Note that in case of external electric fields, which only depend on time, the variety of possible single-pair states is largely reduced. As a consequence of the homogeneity of the external electric field and the resulting translational invariance of the Hamiltonian, only single-pair states where the electron and the positron have the same kinetic energy and opposite momentum are created. This also significantly restricts what kind of multi-pair states can be created. For space-dependent fields, however, single-pair states with different kinetic energies for the electron and the positron are possible and, hence, allowed compositions of multi-pair states are mainly restricted by the Pauli exclusion principle when combining single-pair states into multi-pair states.

The amplitudes $c_{(m,n)}$ can be utilized to determine various observables of the multi-pair states, e. g., the average total spin (11) and helicity (12). Depending on whether the counter-propagating electromagnetic fields have the same or opposite helicities two different symmetries can be found. For setups with the same nonzero helicities ($\alpha_+ = \pi/2 - \alpha_-$, $\alpha_{\pm} \neq \pi/4$), there is a nonzero averaged helicity h_N^{\pm} for the multi-pair states, as shown in Fig. 3. The dynamics of h_N^{\pm} possess a sudden change from positive to negative around an interaction time of 200 laser cycles. This can be attributed to the evolution of the single-pair coefficients plotted in Fig. 2(a). Around 200 laser cycles, states with negative helicity (green dashed line in Fig. 2(a)) obtain a larger weight compared to states with positive helicity. For the used setup with $\alpha_+ = 0.2\pi/4$, both counterpropagating laser fields' helicities are positive but not maximal. Each of these plane waves can be interpreted as a superposition of an intense circular laser field with positive helicity and a less intense circular laser field with negative helicity. Due to the presence of this negative-helicity field, also multi-pair states with negative helicity are produced and their quantum probabilities are higher than those of states with positive helicity at around 200 laser cycles. Note that the definition of the averaged helicity (12) considers electrons and positrons separately. Due to symmetry reasons the average helicities for

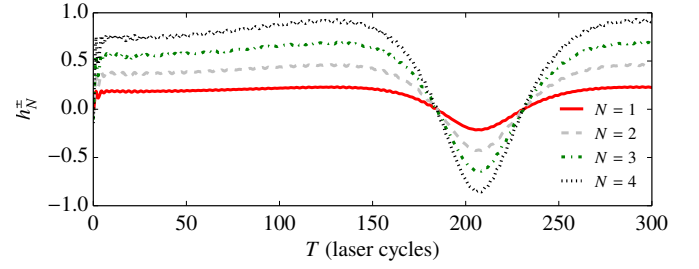


FIG. 3: Time evolution of the average total helicity (12) of the produced electrons and positrons in different multi-pair sectors. Laser parameters are as in Fig. 2.

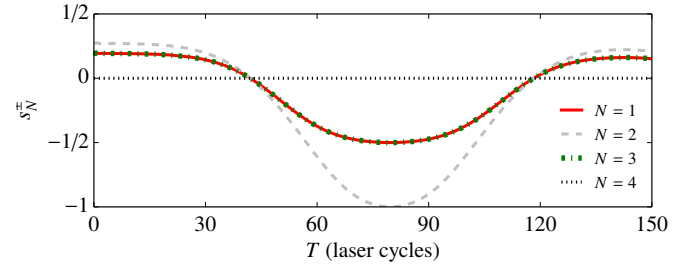


FIG. 4: Time evolution of the averaged total spin (11) for the produced electrons and positrons in different multi-pair sectors. Nonzero spin occurs only, if both laser fields have opposite helicity and are not exactly linearly polarized. Laser field parameters are $\omega = 0.4715m_0$, $E = 3.1 \times 10^{17}$ V/m, and $\alpha_+ = \alpha_- = 0.7\pi/4$.

electrons and positrons yield the same values. Furthermore, the average total spin (11) is zero because the electromagnetic field has no preferred spin direction in this case.

Analogously, if both laser beams have opposite nonzero helicity, the averaged total helicity of the created multi-pair states vanishes. Due to the now preferred spin direction of the laser field, the averaged total spin s_N^{\pm} , defined in Eq. (11), can take nonzero values which in turn may lead to spin-polarized electron-positron pairs [13, 69]. The averaged spin changes its sign depending on the interaction duration for the same reasons given above for the helicity, see Fig. 4. The averaged spin values for the $N = 3$ multi-pair sector are the same as for the single-pair states with $N = 1$ and, furthermore, the averaged spin values for the $N = 4$ sector is exactly zero. Both peculiarities can be explained by the constraints due to the Pauli principle when combining single-pair states into multi-pair states. For example, all permissible 4-pair states consist of exactly two pairs with the electron and the positron having spin up, and two pairs with the electron and the positron having spin down, which yields a zero averaged spin value. A hypothetical state of $N = 4$ pairs with a nonzero total spin, where 3 or 4 of its single pairs have the same spin for electron (positron), is not allowed because at least one of the single pairs would share the same quantum numbers with another pair.

5. Conclusions

A quantum field theoretical treatment of the matter field has been put forward, which yields by numerical means the fermionic quantum field state that results after the action of a strong electromagnetic field. It contains all information about the spectrum of created pairs, including single pairs but also multi-pair configurations. This quantum field state can be determined by solving numerically the time-dependent Dirac equation for all basis vectors of the corresponding Hilbert space. This approach opens the door to study the complex dynamics of pair creation including the creation of multi-electron-positron states and to characterize also correlations between various multi-pair states. It is particularly useful for studying nonperturbative multiphoton pair creation at $\xi \approx 1$. It is, however, neither restricted to $\xi \approx 1$ nor to $\omega \approx m_0$, which was chosen here to keep the computational demand small. Furthermore, the combination of high-intensity lasers with accelerated electrons may produce effective field strengths close to the critical field [16, 17, 30], which is a scenario that can also be treated by the approach as proposed here with minor modifications.

A striking feature of this method is that it yields the probabilities of detecting specific multi-pair states and it shows how these probabilities evolve during the course of interaction with the electromagnetic environment. Furthermore, the polarization and symmetry of the classical laser field persist in the quantum realm for the produced particles. Finally, also pure quantum effects like the Pauli exclusion become apparent.

Acknowledgements

We acknowledge helpful discussions with Antonino Di Piazza and Sebastian Meuren.

-
- [1] F. Sauter, Über das Verhalten eines Elektrons im homogenen elektrischen Feld nach der relativistischen Theorie Diracs, *Z. Physik* 69 (11–12) (1931) 742–764. [doi:10.1007/bf01339461](#).
 - [2] W. Heisenberg, H. Euler, Folgerungen aus der Diracschen Theorie des Positrons, *Z. Physik* 98 (11–12) (1936) 714–732. [doi:10.1007/bf01343663](#).
 - [3] W. Dittrich, M. Reuter, Effective Lagrangians in Quantum Electrodynamics, Vol. 220 of Lecture Notes in Physics, Springer, Heidelberg, 1985. [doi:10.1007/3-540-15182-6](#).
 - [4] Y. Kluger, J. M. Eisenberg, B. Svetitsky, F. Cooper, E. Mottola, Fermion pair production in a strong electric field, *Phys. Rev. D* 45 (12) (1992) 4659–4671. [doi:10.1103/PhysRevD.45.4659](#).
 - [5] S. Schmidt, D. Blaschke, G. Röpke, S. A. Smolyansky, A. V. Prozorkevich, V. D. Toneev, A quantum kinetic equation for particle production in the Schwinger mechanism, *Int. J. Mod. Phys. E* 07 (06) (1998) 709–722. [doi:10.1142/S0218301398000403](#).
 - [6] M. Ruf, G. R. Mocken, C. Müller, K. Z. Hatsagortsyan, C. H. Keitel, Pair production in laser fields oscillating in space and time, *Phys. Rev. Lett.* 102 (8) (2009) 080402. [doi:10.1103/physrevlett.102.080402](#).
 - [7] T. Cheng, Q. Su, R. Grobe, Creation of multiple electron-positron pairs in arbitrary fields, *Phys. Rev. A* 80 (1) (2009) 013410. [doi:10.1103/physreva.80.013410](#).
 - [8] S. S. Bulanov, V. D. Mur, N. B. Narozhny, J. Nees, V. S. Popov, Multiple colliding electromagnetic pulses: A way to lower the threshold of $e^+ e^-$ pair production from vacuum, *Phys. Rev. Lett.* 104 (22) (2010) 220404. [doi:10.1103/physrevlett.104.220404](#).
 - [9] A. Di Piazza, C. Müller, K. Z. Hatsagortsyan, C. H. Keitel, Extremely high-intensity laser interactions with fundamental quantum systems, *Rev. Mod. Phys.* 84 (3) (2012) 1177–1228. [doi:10.1103/revmodphys.84.1177](#).
 - [10] C. Kohlfürst, H. Gies, R. Alkofer, Effective mass signatures in multiphoton pair production, *Phys. Rev. Lett.* 112 (5) (2014) 050402. [doi:10.1103/physrevlett.112.050402](#).
 - [11] O. J. Pike, F. Mackenroth, E. G. Hill, S. J. Rose, A photon-photon collider in a vacuum hohlraum, *Nat. Photon.* 8 (2014) 434–436. [doi:10.1038/nphoton.2014.95](#).
 - [12] V. Kasper, F. Hebenstreit, J. Berges, Fermion production from real-time lattice gauge theory in the classical-statistical regime, *Phys. Rev. D* 90 (2) (2014) 025016. [doi:10.1103/PhysRevD.90.025016](#).
 - [13] A. Wöller, H. Bauke, C. H. Keitel, Spin polarized electron-positron pair production via elliptical polarized laser fields, *Phys. Rev. D* 91 (12) (2015) 125026. [doi:10.1103/physrevd.91.125026](#).
 - [14] S. Meuren, K. Z. Hatsagortsyan, C. H. Keitel, A. Di Piazza, High-energy recollision processes of laser-generated electron-positron pairs, *Phys. Rev. Lett.* 114 (14) (2015) 143201. [doi:10.1103/physrevlett.114.143201](#).
 - [15] I. Ahmad, S. M. Austin, B. B. Back, R. R. Betts, F. P. Calaprice, K. C. Chan, A. Chishti, C. Conner, R. W. Dunford, J. D. Fox, et al., Search for monoenergetic positron emission from heavy-ion collisions at Coulomb-barrier energies, *Phys. Rev. Lett.* 78 (4) (1997) 618–621. [doi:10.1103/physrevlett.78.618](#).
 - [16] D. L. Burke, R. C. Field, G. Horton-Smith, J. E. Spencer, D. Walz, S. C. Berridge, W. M. Bugg, K. Shmakov, A. W. Weidemann, C. Bula, K. T. McDonald, E. J. Prebys, C. Bamber, S. J. Boege, T. Koffas, T. Kotseroglou, A. C. Melissinos, D. D. Meyerhofer, D. A. Reis, W. Ragg, Positron production in multiphoton light-by-light scattering, *Phys. Rev. Lett.* 79 (9) (1997) 1626–1629. [doi:10.1103/physrevlett.79.1626](#).
 - [17] C. Bamber, S. J. Boege, T. Koffas, T. Kotseroglou, A. C. Melissinos, D. D. Meyerhofer, D. A. Reis, W. Ragg, C. Bula, K. T. McDonald, et al., Studies of nonlinear QED in collisions of 46.6 GeV electrons with intense laser pulses, *Phys. Rev. D* 60 (9) (1999) 092004. [doi:10.1103/physrevd.60.092004](#).
 - [18] F. Ehlotzky, K. Krajewska, J. Z. Kamiński, Fundamental processes of quantum electrodynamics in laser fields of relativistic power, *Rep. Prog. Phys.* 72 (2009) 046401. [doi:10.1088/0034-4885/72/4/046401](#).
 - [19] M. Altarelli, R. Brinkmann, M. Chergui, W. Decking, B. Dobson, S. Düsterer, G. Grübel, W. Graeff, H. Graafsma, J. Hajdu, J. Marangos, J. Pflüger, H. Redlin, D. Riley, I. Robinson, J. Rossbach, A. Schwarz, K. Tiedtke, T. Tschentscher, I. Vartanians, H. Wabnitz, H. Weise, R. Wichmann, K. Witte, A. Wolf, M. Wulff, M. Yurkov (Eds.), The European X-Ray Free-Electron Laser Technical Design Report, DESY XFEL Project Group, European XFEL Project Team, Deutsches Elektronen-Synchrotron Member of the Helmholtz Association, Hamburg, 2006. [doi:10.3204/DESY_06-097](#).
 - [20] V. Yanovsky, V. Chvykov, G. Kalinchenko, P. Rousseau, T. Planchon, T. Matsuoka, A. Maksimchuk, J. Nees, G. Cheriaux, G. Mourou, K. Krushelnick, Ultra-high intensity- 300-TW laser

- at 0.1 Hz repetition rate, *Opt. Express* 16 (3) (2008) 2109. [doi:10.1364/OE.16.002109](#).
- [21] B. W. J. McNeil, N. R. Thompson, X-ray free-electron lasers, *Nat. Photon.* 4 (12) (2010) 814–821. [doi:10.1038/nphoton.2010.239](#).
- [22] P. Emma, R. Akre, J. Arthur, R. Bionta, C. Bostedt, J. Bozek, A. Brachmann, P. Bucksbaum, R. Coffee, F.-J. Decker, Y. Ding, D. Dowell, S. Edstrom, A. Fisher, J. Frisch, S. Gilevich, J. Hastings, G. Hays, P. Hering, Z. Huang, R. Iverson, H. Loos, M. Messerschmidt, A. Miahnahri, S. Moeller, H.-D. Nuhn, G. Pile, D. Ratner, J. Rzepiela, D. Schultz, T. Smith, P. Stefan, H. Tompkins, J. Turner, J. Welch, W. White, J. Wu, G. Yocky, J. Galayda, First lasing and operation of an ångström-wavelength free-electron laser, *Nat. Photon.* 4 (9) (2010) 641–647. [doi:10.1038/nphoton.2010.176](#).
- [23] G. A. Mourou, N. J. Fisch, V. M. Malkin, Z. Toroker, E. A. Khazanov, A. M. Sergeev, T. Tajima, B. Le Garrec, Exawatt-zettawatt pulse generation and applications, *Opt. Commun.* 285 (5) (2012) 720–724. [doi:10.1016/j.optcom.2011.10.089](#).
- [24] A. Ringwald, Pair production from vacuum at the focus of an x-ray free electron laser, *Phys. Lett. B* 510 (1–4) (2001) 107–116. [doi:10.1016/s0370-2693\(01\)00496-8](#).
- [25] R. Alkofer, M. B. Hecht, C. D. Roberts, S. M. Schmidt, D. V. Vinnik, Pair creation and an x-ray free electron laser, *Phys. Rev. Lett.* 87 (19) (2001) 193902. [doi:10.1103/physrevlett.87.193902](#).
- [26] N. B. Narozhny, S. S. Bulanov, V. D. Mur, V. S. Popov, On e^+e^- pair production by colliding electromagnetic pulses, *JETP Lett.* 80 (6) (2004) 382–385. [doi:10.1134/1.1830652](#).
- [27] D. B. Blaschke, A. V. Prozorkevich, C. D. Roberts, S. M. Schmidt, S. A. Smolyansky, Pair production and optical lasers, *Phys. Rev. Lett.* 96 (14) (2006) 140402. [doi:10.1103/physrevlett.96.140402](#).
- [28] A. R. Bell, J. G. Kirk, Possibility of prolific pair production with high-power lasers, *Phys. Rev. Lett.* 101 (20) (2008) 200403. [doi:10.1103/physrevlett.101.200403](#).
- [29] D. B. Blaschke, A. V. Prozorkevich, G. Röpke, C. D. Roberts, S. M. Schmidt, D. S. Shkirmanov, S. A. Smolyansky, Dynamical Schwinger effect and high-intensity lasers. Realising nonperturbative QED, *Eur. Phys. J. D* 55 (2) (2009) 341–358. [doi:10.1140/epjd/e2009-00156-y](#).
- [30] G. V. Dunne, New strong-field QED effects at extreme light infrastructure, *Eur. Phys. J. D* 55 (2) (2009) 327–340. [doi:10.1140/epjd/e2009-00022-0](#).
- [31] A. M. Fedotov, N. B. Narozhny, G. Mourou, G. Korn, Limitations on the attainable intensity of high power lasers, *Phys. Rev. Lett.* 105 (2010) 080402. [doi:10.1103/PhysRevLett.105.080402](#).
- [32] S. S. Bulanov, T. Z. Esirkepov, A. G. R. Thomas, J. K. Koga, S. V. Bulanov, Schwinger limit attainability with extreme power lasers, *Phys. Rev. Lett.* 105 (22) (2010) 220407. [doi:10.1103/physrevlett.105.220407](#).
- [33] N. V. Elkina, A. M. Fedotov, I. Y. Kostyukov, M. V. Legkov, N. B. Narozhny, E. N. Nerush, H. Ruhl, QED cascades induced by circularly polarized laser fields, *Phys. Rev. ST Accel. Beams* 14 (2011) 054401. [doi:10.1103/PhysRevSTAB.14.054401](#).
- [34] C. P. Ridgers, C. S. Brady, R. Duclous, J. G. Kirk, K. Bennett, T. D. Arber, A. P. L. Robinson, A. R. Bell, Dense electron-positron plasmas and ultraintense γ rays from laser-irradiated solids, *Phys. Rev. Lett.* 108 (16) (2012) 165006. [doi:10.1103/physrevlett.108.165006](#).
- [35] B. King, N. Elkina, H. Ruhl, Photon polarization in electron-seeded pair-creation cascades, *Phys. Rev. A* 87 (2013) 042117. [doi:10.1103/PhysRevA.87.042117](#).
- [36] A. A. Mironov, N. B. Narozhny, A. M. Fedotov, Collapse and revival of electromagnetic cascades in focused intense laser pulses, *Phys. Lett. A* 378 (44) (2014) 3254–3257. [doi:10.1016/j.physleta.2014.09.058](#).
- [37] V. F. Bashmakov, E. N. Nerush, I. Y. Kostyukov, A. M. Fedotov, N. B. Narozhny, Effect of laser polarization on quantum electrodynamical cascading, *Phys. Plasmas* 21 (1) (2014) 013105. [doi:10.1063/1.4861863](#).
- [38] A. Gonoskov, S. Bastrakov, E. Efimenko, A. Ilderton, M. Marklund, I. Meyerov, A. Muraviev, A. Sergeev, I. Surmin, E. Wallin, Extended particle-in-cell schemes for physics in ultrastrong laser fields: Review and developments, *Phys. Rev. E* 92 (2015) 023305. [doi:10.1103/PhysRevE.92.023305](#).
- [39] E. Brezin, C. Itzykson, Pair production in vacuum by an alternating field, *Phys. Rev. D* 2 (7) (1970) 1191–1199. [doi:10.1103/physrevd.2.1191](#).
- [40] A. Wöller, M. Klaiber, H. Bauke, C. H. Keitel, Relativistic tunneling picture of electron-positron pair creation, *Phys. Rev. D* 91 (6) (2015) 065022. [doi:10.1103/physrevd.91.065022](#).
- [41] H. Hu, C. Müller, C. H. Keitel, Complete QED theory of multiphoton trident pair production in strong laser fields, *Phys. Rev. Lett.* 105 (8) (2010) 080401. [doi:10.1103/physrevlett.105.080401](#).
- [42] R. Schützhold, H. Gies, G. V. Dunne, Dynamically assisted Schwinger mechanism, *Phys. Rev. Lett.* 101 (13) (2008) 130404. [doi:10.1103/physrevlett.101.130404](#).
- [43] A. Di Piazza, E. Lötstedt, A. I. Milstein, C. H. Keitel, Barrier control in tunneling e^+e^- photoproduction, *Phys. Rev. Lett.* 103 (17) (2009) 170403. [doi:10.1103/PhysRevLett.103.170403](#).
- [44] A. Otto, D. Seipt, D. Blaschke, B. Kämpfer, S. A. Smolyansky, Lifting shell structures in the dynamically assisted Schwinger effect in periodic fields, *Phys. Lett. B* 740 (2015) 335–340. [doi:10.1016/j.physletb.2014.12.010](#).
- [45] A. Otto, D. Seipt, D. Blaschke, S. A. Smolyansky, B. Kämpfer, Dynamical Schwinger process in a bifrequent electric field of finite duration: Survey on amplification, *Phys. Rev. D* 91 (10) (2015) 105018. [doi:10.1103/PhysRevD.91.105018](#).
- [46] This is very similar to the intermediate domain of strong-field ionization between the tunneling and multiphoton regimes, as studied in [70].
- [47] G. R. Mocken, M. Ruf, C. Müller, C. H. Keitel, Nonperturbative multiphoton electron-positron-pair creation in laser fields, *Phys. Rev. A* 81 (2) (2010) 022122. [doi:10.1103/PhysRevA.81.022122](#).
- [48] K. Krajewska, J. Z. Kamiński, Correlations in laser-induced electron-positron pair creation, *Phys. Rev. A* 84 (3) (2011) 033416. [doi:10.1103/physreva.84.033416](#).
- [49] S. Augustin, C. Müller, Nonperturbative Bethe-Heitler pair creation in combined high- and low-frequency laser fields, *Phys. Lett. B* 737 (2014) 114–119. [doi:10.1016/j.physletb.2014.08.042](#).
- [50] I. Akal, S. Villalba-Chávez, C. Müller, Electron-positron pair production in a bifrequent oscillating electric field, *Phys. Rev. D* 90 (11) (2014) 113004. [doi:10.1103/physrevd.90.113004](#).
- [51] Z. L. Li, D. Lu, B. S. Xie, B. F. Shen, L. B. Fu, J. Liu, Nonperturbative signatures in pair production for general elliptic polarization fields, *Eur. Phys. Lett.* 110 (5) (2015) 51001. [doi:10.1209/0295-5075/110/51001](#).
- [52] J. Schwinger, On gauge invariance and vacuum polarization, *Phys. Rev.* 82 (5) (1951) 664–679. [doi:10.1103/PhysRev.82.664](#).
- [53] F. Hebenstreit, R. Alkofer, G. V. Dunne, H. Gies, Quantum statistics effects for Schwinger pair production in short laser pulses, in: K. A. Milton, M. Bordag (Eds.), *Proceedings of the Ninth*

- Conference on Quantum Field Theory Under the Influence of External Conditions (QFEXT09), World Scientific, 2010, pp. 492–496. [doi:10.1142/9789814289931_0062](https://doi.org/10.1142/9789814289931_0062).
- [54] S. Schmidt, D. Blaschke, G. Röpke, A. V. Prozorkevich, S. A. Smolyansky, V. D. Toneev, Non-markovian effects in strong-field pair creation, *Phys. Rev. D* 59 (9) (1999) 094005. [doi:10.1103/PhysRevD.59.094005](https://doi.org/10.1103/PhysRevD.59.094005).
- [55] F. Hebenstreit, J. Berges, D. Gelfand, Simulating fermion production in 1 + 1 dimensional QED, *Phys. Rev. D* 87 (10) (2013) 105006. [doi:10.1103/physrevd.87.105006](https://doi.org/10.1103/physrevd.87.105006).
- [56] E. S. Fradkin, D. M. Gitman, Furry picture for quantum electrodynamics with pair-creating external field, *Fortschr. Phys.* 29 (9) (1981) 381–411. [doi:10.1002/prop.19810290902](https://doi.org/10.1002/prop.19810290902).
- [57] E. S. Fradkin, D. M. Gitman, S. M. Shvartsman, *Quantum Electrodynamics with Unstable Vacuum*, Springer Series in Nuclear and Particle Physics, Springer, Heidelberg, 1991.
- [58] J. W. Braun, Q. Su, R. Grobe, Numerical approach to solve the time-dependent Dirac equation, *Phys. Rev. A* 59 (1) (1999) 604–612. [doi:10.1103/physreva.59.604](https://doi.org/10.1103/physreva.59.604).
- [59] H. Bauke, C. H. Keitel, Accelerating the Fourier split operator method via graphics processing units, *Comp. Phys. Commun.* 182 (12) (2011) 2454–2463. [doi:10.1016/j.cpc.2011.07.003](https://doi.org/10.1016/j.cpc.2011.07.003).
- [60] F. Fillion-Gourdeau, E. Lorin, A. D. Bandrauk, Numerical solution of the time-dependent Dirac equation in coordinate space without fermion-doubling, *Comp. Phys. Commun.* 183 (7) (2012) 1403–1415. [doi:10.1016/j.cpc.2012.02.012](https://doi.org/10.1016/j.cpc.2012.02.012).
- [61] N. B. Narozhny, A. M. Fedotov, Extreme light physics, *Contemp. Phys.* 56 (3) (2015) 249–268. [doi:10.1080/00107514.2015.1028768](https://doi.org/10.1080/00107514.2015.1028768).
- [62] Units, where the speed of light and the reduced Planck constant are set to unity, are employed.
- [63] F. Gross, *Relativistic Quantum Mechanics and Field Theory*, Wiley-VCH, Weinheim, 2004.
- [64] B. Thaller, *Advanced Visual Quantum Mechanics*, Springer, Heidelberg, 2005.
- [65] P. Krekora, Q. Su, R. Grobe, Klein paradox in spatial and temporal resolution, *Phys. Rev. Lett.* 92 (2004) 040406. [doi:10.1103/PhysRevLett.92.040406](https://doi.org/10.1103/PhysRevLett.92.040406).
- [66] T. Cheng, Q. Su, R. Grobe, Introductory review on quantum field theory with space-time resolution, *Contemp. Phys.* 51 (4) (2010) 315–330. [doi:10.1080/00107510903450559](https://doi.org/10.1080/00107510903450559).
- [67] V. P. Frolov, D. M. Gitman, Density matrix in quantum electrodynamics, equivalence principle and Hawking effect, *J. Phys. A: Math. Gen.* 11 (7) (1978) 1329. [doi:10.1088/0305-4470/11/7/019](https://doi.org/10.1088/0305-4470/11/7/019).
- [68] R. C. Jones, A new calculus for the treatment of optical systems, *J. Opt. Soc. Am.* 31 (7) (1941) 488–493. [doi:10.1364/josa.31.000488](https://doi.org/10.1364/josa.31.000488).
- [69] E. Strobel, S.-S. Xue, Semiclassical pair production rate for rotating electric fields, *Phys. Rev. D* 91 (4) (2015) 045016. [doi:10.1103/PhysRevD.91.045016](https://doi.org/10.1103/PhysRevD.91.045016).
- [70] M. Klaiber, K. Z. Hatsagortsyan, C. H. Keitel, Tunneling dynamics in multiphoton ionization and attoclock calibration, *Phys. Rev. Lett.* 114 (8) (2015) 083001. [doi:10.1103/physrevlett.114.083001](https://doi.org/10.1103/physrevlett.114.083001).